APPLICATIONS OF SPECTRAL GRAPH THEORY IN THE FIELD OF TELECOMMUNICATIONS

ПРИЛОЖЕНИЕ НА СПЕКТРАЛНАТА ТЕОРИЯ НА ГРАФИТЕ В ТЕЛЕКОМУНИКАЦИИТЕ

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Ключови думи: мрежови топологии, графи, устойчивост на мрежа, спектрална теория на графи

Развитието на телекомуникационните мрежи налага да се взимат предвид множество аспекти. При първоначално проектиране и последваща оптимизация, основен аспект на телекомуникационните мрежи е тяхната топология. Протоколите за динамична маршрутизация позволяват изграждане, успешно управление и използване на глобални по мащаб и много разнообразни по вид и топология мрежи. Обикновено ограниченията за топологията на мрежата са от географско, икономическо и технологично естество. Остават, обаче, открити въпросите свързани с основната топология на мрежата - Как да бъде изградена?, Как да се осигури необходимата резервираност и устойчивост на мрежата? и др. Тази разработка има за цел да разгледа основните спектрални параметри на графите, които допринасят за определянето на устойчивостта и резервираността на мрежите.

Keywords: network topology, graphs, network resilience, spectral graph theory

The quick expansion of the networks worldwide, leads to number of question regarding their topology. In the initial planning and further optimizations the network topology is a key factor. The dynamic routing protocols give the ability to have global networks implementing a variety of complex topologies. Usually constrains for choosing a network topology are geographical, economical and technical. However often disregarded questions is about now to choose a topology – Which one to choose initially?, How the ensure the needed redundancy and resilience of the network?, etc. This paper reviews the basic spectral parameters of graphs that define the resilience and redundancy of networks.

INTRODUCTION

Graph theory has always been used for solving a number of problems in the field of Telecommunications. As each network can be represented as graph – either weighted or unweighted, either undirected, or directed – the graph theory can be used to find paths in the network, to calculate the shortest one or to search for critical edges (bridges) or critical nodes – called articulation points. Until recently most of these tasks were accomplished algorithmically. In the recent decades a new branch of graph theory has emerged – Spectral graph theory [1]. Its foundations lie on the idea of representing the graph in matrix form – either adjacency matrix, or other type – and calculating the parameters of the graph based on the parameters of the matrices.

Usually these parameters give a lot of information on the graph structure and its properties [2], both local and global.

Spectral graph theory is a mathematical theory in which linear algebra and graph theory meet. For any graph matrix *M* we can build a spectral graph theory in which graphs are studied by means of eigenvalues of the matrix *M*. This theory is called *M*-theory.

In order to avoid confusion, to any notion in this theory a prefix *M*-could be added (e.g., *M*-eigenvalues). Frequently used graph matrices are the adjacency matrix A, the Laplacian $L = D - A$ and the signless Laplacian $Q = D + A$, where D is a diagonal matrix of vertex degrees. *The* Spectral graph theory includes all particular theories together with interaction tools.

GRAPH THEORY APPLICATION IN THE FIELD OF TELECOMMUNICATIONS

Since the beginning of networks studies the graph theory was used to solve different problems in the topology of networks. As networks grew bigger and more complex, and algorithmic approach was used to solve problems like path search, depth search, etc. The **typical problems** solved by graph theory are:

- **SPT – Shortest path tree** [3]

This algorithm is used to solve the problem of constructing the tree of minimum total length between the n nodes. Now it and its modification for weighted graphs (Bellman–Ford–Moore) [4] is largely used in link state protocols (like OSPF).

- **DFS - Depth-first search**

This algorithm is for traversing or searching tree or graph data structures. It starts at the root (selecting some arbitrary node as the root in the case of a graph) and explores as far as possible along each branch before backtracking. The first depth-first search was investigated in the 19th century by French mathematician Charles Pierre Trémaux as a strategy for solving mazes. Now it is largely used for finding connected (and strongly connected) components, in finding the bridges of a graph (see below), and others.

The computational complexity of DFS was investigated by Reif, who showed that a decision version of it (establish whether some vertex *u* occurs before some vertex *v* in a DFS order of a rooted graph) is *P*-complete [5], meaning that it is "a nightmare for parallel processing" [6].

- **BFS - Breadth-first search**

The Breadth-first search (BFS) is an algorithm for traversing or searching tree or graph data structures. It starts at the tree root (or some arbitrary node of a graph) and explores the neighbour nodes first, before moving to the next level neighbours. .

BFS was invented in the late 1950s by E. F. Moore, who used it to find the shortest path out of a maze, and discovered independently by C. Y. Lee as a wire routing algorithm (published 1961) [7]. It is used for Finding the shortest path between two nodes u and v (with path length measured by number of edges) in distance vector and path vector protocols, in the Ford–Fulkerson method for computing the maximum flow in a flow network and in other methods, not directly related to telecommunications.

- **Bridges and critical nodes search**

There are a number of methods for finding a graph's articulation points, which differ in terms of complexity and iterations. In [8] a basic algorithm for finding articulation points is defined. In [9] a further more optimal methods are defined and also an *"impact"* factor of each articulation point is defined.

All above applications of graph theory have algorithmic approach and are still used as noted in each. However none of them has the ability to characterize a graph (or network) topology for its global properties and the local properties of each node (besides its degree, connectivity and *impact* factor).

SPECTRAL GRAPH THEORY

Although above mentioned algorithms are powerful tools, they are not sufficient for rigorous analysis of network topologies. Therefore it is needed to calculate the graph metrics of networks and critical infrastructures [10].

Some of the well-known metrics provide insight on a variety of graph properties, including distance, degree of connectivity, and centrality. Network diameter, radius, and average hop count provide distance measures [11],[12]. Eccentricity of a node is the longest shortest path from this node to every other node; the largest value of eccentricity among all nodes is the diameter and the smallest eccentricity is the radius. Betweenness is the number of shortest paths through a node or link and provides a centrality or importantness measure [11]. Clustering coefficient is a centrality measure of how well a node's neighbours are connected [11]. Closeness centrality is the inverse of the sum of shortest paths from a node to every other node [13]. Algebraic connectivity [14], *a(G)*, is the second smallest eigenvalue of the Laplacian matrix. For the graphs of the same order (number of vertices), algebraic connectivity provides a very good measure of how well the graph is connected and it indicates robustness of networks against node and link failures [15].

The field of study of the spectral graph theory are the parameters of a graph that are in relation to the characteristics of its associated matrices, such as its adjacency matrix or the Laplacian matrix.

Definitions:

Let $G_c = (V, E)$ be a weighted graph on n vertices in which the edge weights are positive numbers. Denote by $i \sim j$ if the vertices i and j are adjacent and by $w_{i,j}$ the weight of the edge $e_{i,j}$. Let $w_i = \sum_{j \sim i} w_{i,j}$. The Laplacian matrix $L(G_C) = (l_{i,j})$ of G_c is defined as

$$L_{i,j} = \begin{cases} w_i, & \text{if } i = j, \\ -w_{i,j} & \text{if } i \sim j, \\ 0, & \text{otherwise.} \end{cases}$$

Note that G_c is a simple graph whenever $w_{i,j} = 1$ for all $i \sim j$. Denote a simple graph by G.

The $L(G_C)$ is a real symmetric matrix. From this fact and Gershgorin circle theorem, it follows that its eigenvalues are nonnegative real numbers. Moreover, since its rows sum to *0*, *0* is the smallest eigenvalue of $L(G_C)$ and the multiplicity of *0* equals the number of components of G_c. Then we can assume that G_c is a simple connected graph.

Also let T denote the diagonal matrix with *(i,i)*-th entry having w_i. The normalized Laplacian matrix of $G(\mathcal{L})$ is defined to be the matrix:

$$\mathcal{L}_{i,j} = \begin{cases} 1, & \text{if } i = j \text{ and d} \neq 0; \\ -\frac{1}{\sqrt{d_i d_j}}, & \text{if } i \sim j; \\ 0, & \text{otherwise.} \end{cases}$$

We can then write

$$\mathcal{L}_{i,j} = T^{-\frac{1}{2}} L T^{-\frac{1}{2}}$$

with the convention $T^{-1}(i,i) = 0$ for $d_i = 0$

From these definitions we can retrieve the important parameters of graph spectra:

- **Diameter**

The diameter is not a specific spectral parameter of a graph, but is used a lot in spectral theory. The diameter of a graph is defined as the longest shortest-path between any two graph vertices (u,v) of a graph:

$$d_G = \max_{i,j} l_{i,j}$$

This implies the use of BFS algorithm to travel through the graph to find all paths.[16]

It also can be defined as the maximum eccentricity among the vertices of G, as follows:

$$d_G = \max_{v \in V} \epsilon(v)$$

The diameter of graph is fundamental parameter of the networks, as it gives quick estimation of the size and worst case of paths through the network.

- **Spectrum of a graph**

The spectrum of finite graph G_c is by definition the spectrum of the adjacency matrix A, its set of eigenvalues together with their multiplicities. The Laplace spectrum of finite graph G_c is the spectrum of the Laplace matrix L[17].

Since A is real and symmetric, all its eigenvalues are real. Also, for each eigenvalue λ_n, its algebraic multiplicity coincides with its geometric multiplicity. Since A has zero diagonal, its trace *tr(A)*, and hence the sum of the eigenvalues is zero.

Similarly, L is real and symmetric, so that the Laplace spectrum is real. Moreover, L is positive semidefinite and singular, so we can denote the eigenvalues by:

$$\lambda_n \geq \cdots \geq \lambda_2 \geq \lambda_1 = 0$$

The sum of these eigenvalues is *tr(L),* which is twice the number of edges of G_c. Finally, also $\mathcal{L}$ has real spectrum and nonnegative eigenvalues (but not necessarily singular) and *tr(*$\mathcal{L}$*)=tr(L).*

- **Algebraic connectivity**

In [14] the algebraic connectivity $a(G_C)$ of a (connected) graph is defined as the second smallest eigenvalue (λ_2) of the Laplacian matrix of a graph with *n* vertices.

This parameter is used as a generalized measure of “how well is the graph connected” [15]. It has values between *0* and *n* (a fully-connected graph K_n has *n*). This eigenvalue is greater than 0 if and only if G is a connected graph. This is a corollary to the fact that the number of times 0 appears as an eigenvalue in the eigenvector of the Laplacian is the number of connected components in the graph. Therefore, the farther λ_2 is from zero, the more difficult it is to separate a graph into independent components. However, the algebraic connectivity is equal to zero for all disconnected networks. Therefore, as soon as the connectedness is lost, due to failures for example, this measure becomes less useful by being too coarse.

- **Effective Network resistance**

A good measure for network robustness is the effective resistance. The normalized total effective resistance is proportional to the inverse total effective resistance, which is defined as the sum of the pairwise effective resistances over all pairs of vertices. [18] The total effective resistance R^{tot} is the sum of the effective resistances over all pairs of vertices, where the effective resistance of the edges is defined as:

$$R_{ab} = \frac{v_a - v_b}{Y}$$

And the total effective resistance is:

$$R^{tot} = \sum_{i=1}^{n} \sum_{j=i+1}^{n} R_{ij}$$

In the literature the total effective resistance is also called Kirchhoff index. As a result of Klein and Randić work[19], it can be written as a function of the non-zero Laplacian (weighted) eigenvalues:

$$R^{tot} = n \sum_{i=2}^{n} \frac{1}{\lambda_i}$$

For network robustness index the value of normalized effective resistance is used. The advantage of this is that values lie in the interval *[0;1]* and a large value indicates a robust network. It is zero for unconnected graphs and maximal (one) for complete graphs – similar to the algebraic connectivity:

$$R^{norm} = \frac{n-1}{R^{tot}} = \frac{n-1}{n \sum_{i=2}^{n} \frac{1}{\lambda_i}} \in [0,1]$$

In [20], [21] the term **Network criticality** is used. It is defined as the average random-walk betweenness of a link (node) normalized by its weight. This quantity is independent of link (node) location and it is a decreasing and strictly convex function of link weights. Network criticality can be written in terms of the components of the undirected Moore-Penrose Laplacian matrix:

$$\hat{\tau} = \frac{2}{n-1} tr(L^{+})$$

There is a useful interpretation of network criticality in terms of electrical circuits: network criticality is the unweighted average of the equivalent resistances. Therefore optimizing criticality is equivalent to minimizing the average resistance or maximizing the average conductance of a network, which explains why network criticality can be considered also as a global robustness metric.

Relations between graph spectral parameters

Although there is no 1-to-1 relationship between the specified above parameter, there are several inequalities that can be used to quickly evaluate the degree of other parameter. One of the most useful inequality is to estimate the diameter of a graph, based on its spectral properties, as the usual way to evaluate the diameter is find all SPFs which, for larger graphs, can be time-consuming.

In [22] imposes an upper bound on the diameter of G based on the eigenvalue λ_2

$$\text{diam(G)} \leq 2 \left[\frac{\Delta+\lambda_2}{4\lambda_2} \ln(\text{n-1}) \right]$$

A lower bound is also defined as:

$$\text{diam}(G) \geq \frac{4}{n\lambda_2}$$

These bounds, however, are usually not strong enough, and depending on the graph type, there might be better inequalities.

The mean distance $\bar{\rho}G)$ of G is equal to the average of all distances between distinct vertices of G:

$$\bar{\rho}(G) \leq \frac{n}{n-1} \left[\left(\frac{\Delta+\lambda_2}{4\lambda_2} \ln(n-1) \right) + \frac{1}{2} \right]$$

Also there is a lower limit for the mean distance [23]:

$$(n-1)\bar{\rho}(G) \geq \frac{2}{\lambda_2} + \frac{n-2}{2}$$

The network criticality is bounded by the reciprocal of algebraic connectivity:

$$\frac{2}{(n-1)\lambda_2} \leq \hat{\tau} \leq \frac{2}{\lambda_2}$$

One of the future works is to study these bounds and find the relationship between and other eigenvalues of the Laplacian matrix of graphs and specify stronger bounds to the parameters as this will give better optimization scenarios based on optimizing the spectral properties of the graph.

CONCLUSION

Spectral graph theory gives very good methodology for evaluating the topology of networks and their optimization. As all spectral parameters are correlated between each other, so the further works are on in-depth study of all the relationships between these parameters and finding optimal topologies by optimizing the graph spectral characteristics in given bounds.

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